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On p-adic interpolating function for q-Euler numbers and its derivatives

Taekyun Kim

EECS, Kyungpook National University, Taegu 702-701, South Korea

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Abstract

In this paper we study a two-variable p-adic q–l-function $l_{p,q}(s,t|\chi)$ for Dirchlet's character χ, with the property that

$$l_{p,q}(-n,t|\chi)=E_{n,\chi_n,q}(pt)-\frac{[2]_q}{[2]_{q^p}}\chi_n(p)[p]_q^n E_{n,\chi_n,q^p}(t)$$

for positive integers n and $t\in\mathbb{C}_p$ with $|t|_p\leqslant 1$, and $E_{n,\chi_n,q}(x)$ generalized Euler polynomials. Finally, we prove that $l_{p,q}(s,t|\chi)$ is analytic in s and t for $s\in\mathbb{C}_p$ with $|s|_p<p^{1-\frac{1}{p-1}}$ and $t\in\mathbb{C}_p$ with $|t|_p\leqslant 1$.

Keywords: Zeta function; q-Series; p-Adic interpolating functions; q-Euler numbers; Partial zeta function; Analytic function

1. Introduction and preliminaries

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ denote, respectively, the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p=\frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex $q\in\mathbb{C}$, or a p-adic number $q\in\mathbb{C}_p$. If $q\in\mathbb{C}$, we assume that $|q|<1$. If $q\in\mathbb{C}_p$, we assume that $|q-1|_p<p^{-\frac{1}{p-1}}$ so that $q^x=\exp(x\log q)$ for $|x|_p\leqslant 1$. We use the notations as

$$[x]_q=\frac{1-q^x}{1-q}\quad\text{and}\quad[x]_{-q}=\frac{1-(-q)^x}{1+q},\qquad\text{cf. [8,9].}$$

For a fixed positive integer d with $(p,d)=1$, $n\in\mathbb{N}$, set

$$X=X_d=\varprojlim_n \mathbb{Z}/dp^n\mathbb{Z},\qquad X_1=\mathbb{Z}_p,$$

E-mail addresses: tkim64@hanmail.net, tkim@knu.ac.kr.

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$$X^* = \bigcup_{\substack{0<a<dp\\(a,p)=1}} (a+dp\mathbb{Z}_p),$$

$$a+dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^n}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leqslant a < dp^n$.

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{x=0}^{p^n-1} f(x)q^x, \quad \text{cf. [4–6]},$$

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{x=0}^{p^n-1} f(x)(-q)^x, \quad \text{see [14]}.$$

It is well known that the familiar Euler polynomials $E_n(z)$ are defined by means of the following generating function:

$$F(x,t) = \frac{2}{e^t+1}e^{zt} = \sum_{n=0}^{\infty} E_n(z)\frac{t^n}{n!}, \quad \text{see [6–10]}.$$

We note that, by substituting $z=0$, $E_n(0)=E_n$ are the familiar Euler numbers. Over five decades ago, Carlitz defined q-extension of Euler numbers and polynomials, cf. [1–3]. In a recent [5] we defined the new q-extended Euler numbers as follows:

$$E_{0,q} = \frac{[2]_q}{2}, \qquad (qE+1)^n + E_{n,q} = 0 \quad \text{if } n>0, \tag{1}$$

with the usual convention of replacing E^n by $E_{n,q}$. Note that $\lim_{q\to 1} E_{n,q} = E_n$, see [5]. Also, we defined the q-Euler polynomials as

$$\begin{aligned} E_{n,q}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{xl} E_{l,q}[x]_q^{n-l} \\ &= [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{xl}}{1+q^l}. \end{aligned} \tag{2}$$

From (2) we note that

$$E_{n,q} = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l},$$

where $\binom{n}{l} = \frac{n(n-1)\cdots(n-l+1)}{l!}$. Let us consider the generating function of q-Euler polynomials as $F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}$. Then we see that

$$F_q(t,x) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n e^{[n+x]_q t}. \tag{3}$$

Let χ be the Dirichlet's character with conductor d $(=\text{odd}) \in \mathbb{N}$. Then the generalized q-Euler numbers attached to χ, $E_{n,\chi,q}$, are defined by

$$F_{\chi,q}(t) = [2]_q \sum_{n=0}^{\infty} \chi(n)(-1)^n e^{[n]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,q}\frac{t^n}{n!}. \tag{4}$$

By (4), we easily see that

$$\lim_{q\to 1} F_{\chi,q}(t) = \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{at}}{e^{dt}+1} = \sum_{n=0}^{\infty} E_{n,\chi}\frac{t^n}{n!},$$

where $E_{n,\chi}$ are the generalized Euler numbers attached to χ, see [9–12]. Hence,

$$\lim_{q\to 1} E_{n,\chi,q} = E_{n,\chi}.$$

From the simple calculation of (4), we easily derive the following equation:

$$E_{n,\chi,q} = [d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} \chi(a)(-1)^a E_{n,q^d}\left(\frac{a}{d}\right). \tag{5}$$

Now we consider the generalized q-Euler polynomials associated with χ as follows:

$$F_{\chi,q}(x,t) = \sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} \chi(n)(-1)^n e^{[n+x]_q t}. \tag{6}$$

From (6) we note that

$$F_{\chi,q}(x,t) = [2]_q \sum_{a=0}^{d-1} \chi(a)(-1)^a \sum_{n=0}^{\infty}(-1)^n e^{[a+x+nd]_q t}.$$

Hence, we have

$$E_{n,\chi,q}(x) = \sum_{k=0}^{n}\binom{n}{k} q^{kx} E_{k,\chi,q}[x]_q^{n-k}.$$

Since $[a+x+nd]_q = [d]_q[\frac{a+x}{d}+n]_{q^d}$.

In this paper we give a two-variable p-adic q–l-function $l_{p,q}(s,t|\chi)$ for Dirichlet's character χ, with the property that

$$l_{p,q}(-n,t|\chi) = E_{n,\chi_n,q}(pt) - \frac{[2]_q}{[2]_{q^p}}\chi_n(p)[p]_q^n E_{n,\chi_n,q^p}(t),$$

for positive integer n and $t \in \mathbb{C}_p$ with $|t|_p \leqslant 1$, and $E_{n,\chi\omega^{-n}}(x)$ are generalized Euler polynomials. Finally we prove that $l_{p,q}(s,t|\chi)$ is analytic in s and t for $s \in \mathbb{C}_p$ with $|s|_p < p^{1-\frac{1}{p-1}}$ and $t \in \mathbb{C}_p$ with $|t|_p \leqslant 1$.

2. Two-variable Dirichlet's type Euler l-function

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let g $(=\text{odd})$ be a positive integral multiple of d $(=d_x)$. Then for each $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!} &= [2]_q \sum_{a=0}^{g-1}\chi(a)(-1)^a \sum_{k=0}^{\infty} e^{[gk+a+x]_q t}(-1)^k \\ &= [2]_q \sum_{a=0}^{g-1}\chi(a)(-1)^a \sum_{k=0}^{\infty}(-1)^k e^{[g]_q[\frac{x+a}{g}+k]_{q^g} t} \\ &= \sum_{n=0}^{\infty}\left([g]_q^n \frac{[2]_q}{[2]_{q^g}} \sum_{a=0}^{g-1}\chi(a)(-1)^a E_{n,q^g}\left(\frac{x+a}{g}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore we obtain the following lemma.

Lemma 1. *Let g $(=\text{odd})$ be a positive integral multiple of d. Then for each $n \in \mathbb{Z}^+$, we have*

$$E_{n,\chi,q}(x) = [g]_q^n \frac{[2]_q}{[2]_{q^g}} \sum_{a=0}^{g-1} \chi(a)(-1)^a E_{n,q^g}\left(\frac{x+a}{g}\right).$$

Note that the series on the right-hand side of (3) and (6) are uniformly convergent. Hence, we see that

$$E_{k,q}(x) = \frac{d^k}{dt^k} F_q(t,x)\Big|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n [n+x]_q^k$$

and

$$E_{k,\chi,q}(x) = \frac{d^k}{dt^k} F_{\chi,q}(t,x)\Big|_{t=0} = [2]_q \sum_{n=0}^{\infty} \chi(n)(-1)^n [n+x]_q^k, \quad \text{for } k \in \mathbb{Z}_+. \tag{7}$$

For $s \in \mathbb{C}$, we consider Hurwitz's type Euler q-zeta function as follows:

$$\zeta_{q,E}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^k}{[n+x]_q^s}, \quad x \in \mathbb{R}. \tag{8}$$

Note that $\zeta_q(-n,x) = E_{n,q}(x)$ for $n \in \mathbb{Z}_+$. By Lemma 1, we can consider two-variable Dirichlet's type Euler l-series as follows: for $s \in \mathbb{C}$,

$$l_{q,E}(s,x|\chi) = [2]_q \sum_{n=0}^{\infty} \frac{\chi(n)(-1)^n}{[n+x]_q^s}. \tag{9}$$

In [6], one-variable Euler q–l-series is defined as

$$l_{q,E}(s,\chi) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{[n]_q^s}, \quad \text{for } s \in \mathbb{C}.$$

Thus, we note that $l_q(s,0|\chi) = l_q(s,\chi)$. For $0 < x \leqslant 1$, and $s \in \mathbb{C}$, $\zeta_{q,E}(s,x) = \zeta_E(s,x)$ if $q \to 1$, where $\zeta_E(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}$ is the Euler zeta function. It is easy to see that $\zeta_{q,E}(s,x)$ may be continued analytically to the whole complex plane. By (8) and (9), we obtain the following proposition.

Proposition 2. *For $k \in \mathbb{Z}_+$, $x \in \mathbb{R}$ with $0 \leqslant x < 1$, we have*

$$l_{q,E}(-k,x|\chi) = E_{n,\chi,q}(x).$$

Let χ be a Dirichlet's character with conductor d $(=\text{odd}) \in \mathbb{N}$ and let n be an even natural number. Then we have

$$-F_{\chi,q}(nd,t) + F_{\chi,q}(0,t) = \sum_{l=0}^{\infty}\left([2]_q \sum_{k=0}^{nd-1} \chi(k)(-1)^k [k]_q^l\right)\frac{t^l}{l!}.$$

Thus we note that

$$[2]_q \sum_{k=0}^{nd-1} \chi(k)(-1)^k [k]_q^l = -E_{l,\chi,q}(nd) + E_{l,\chi,q}. \tag{10}$$

From (8) and (9) we derive

$$l_{q,E}(s,x|\chi) = [2]_q \sum_{n=0}^{\infty} \frac{\chi(n)(-1)^n}{[n+x]_q^s} = \frac{[2]_q}{[2]_{q^d}} [d]_q^{-s} \sum_{a=1}^{d} \chi(a)(-1)^a \zeta_{q^d,E}\left(s, \frac{a+x}{d}\right). \tag{11}$$

Let $\Gamma(s)$ be the gamma function. By applying Mellin transform to Eq. (6), then we can readily see that

$$\frac{1}{\Gamma(s)}\int_0^\infty F_{\chi,q}(x,-t)t^{s-1}\,dt = l_{q,E}(s,x|\chi), \quad \text{for } s\in\mathbb{C}. \tag{12}$$

Substituting $s=-n$ into (12), we have

$$l_{q,E}(-n,x|\chi) = E_{n,\chi,q}(x), \quad \text{for } n\in\mathbb{Z}_+.$$

Let us consider a partial Euler q-zeta function as follows:

$$\begin{aligned} H_{q,E}(s,a,F) &= \sum_{\substack{m\equiv a \pmod F\\ m>0}} \frac{(-1)^m}{[m]_q^s} = \sum_{n=0}^\infty \frac{(-1)^{a+nF}}{[a+nF]_q^s} \\ &= (-1)^a \sum_{n=0}^\infty \frac{(-1)^n}{[F]_q^s[\frac{a}{F}+n]_{q^F}^s} = (-1)^a \frac{[F]_q^{-s}}{[2]_{q^F}}\left([2]_{q^F}\sum_{n=0}^\infty \frac{(-1)^n}{[\frac{a}{F}+n]_{q^F}^s}\right) \\ &= \frac{[F]_q^{-s}}{[2]_{q^F}}(-1)^a \zeta_{q^F,E}\left(s,\frac{a}{F}\right), \end{aligned} \tag{13}$$

where F $(=\text{odd})$ is positive integers with $0<a<F$. Let χ $(\neq 1)$ be the Dirichlet's character with conductor F $(=\text{odd})$. Then we have

$$l_{q,E}(s,\chi) = [2]_q\sum_{a=1}^F \chi(a)H_{q,E}(s,a,F), \quad \text{for } s\in\mathbb{C}. \tag{14}$$

The function $H_{q,E}(s,a,F)$ is a holomorphic function in whole complex plane. For $a\in\mathbb{Z}_+$, we have

$$H_{q,E}(-n,a,F) = (-1)^a\frac{[F]_q^n}{[2]_{q^F}}E_{n,q^F}\left(\frac{a}{F}\right). \tag{15}$$

We now modify partial Euler q-zeta function as follows:

$$H_{q,E}(s,a,F) = \frac{(-1)^a}{[2]_{q^F}}[a]_q^{-s}\sum_{j=0}^\infty \binom{-s}{j} q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^j E_{j,q^F}, \quad \text{for } s\in\mathbb{C}. \tag{16}$$

From (14) and (16), we note that

$$l_q(s,\chi) = \frac{[2]_q}{[2]_{q^F}}\sum_{a=1}^F (-1)^a\chi(a)[a]_q^{-s}\sum_{j=0}^\infty \binom{-s}{j} q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^j E_{j,q^F}, \quad \text{for } s\in\mathbb{C}. \tag{17}$$

By (9) we see that $l_q(s,x|\chi)$ is an analytic function for $s\in\mathbb{C}$ when $\chi\neq 1$. Using (13) to define $H_{q,E}(s,a+x,F)$ for all $a\in\mathbb{Z}$ with $0<a<F$, $x\in\mathbb{R}$ with $0\leqslant x<1$, we obtain

$$l_q(s,x|\chi) = [2]_q\sum_{a=1}^F \chi(a)H_{q,E}(s,a+x,F). \tag{18}$$

Let F $(=\text{odd})$ and a be positive integers with $0<a<F$. Then we have

$$l_q(s,x|\chi) = \frac{[2]_q}{[2]_{q^F}}\sum_{a=1}^F (-1)^a\chi(a)[a+x]_q^{-s}\sum_{j=0}^\infty \binom{-s}{j} q^{j(a+x)}\left(\frac{[F]_q}{[a+x]_q}\right)^j E_{j,q^F}. \tag{19}$$

From (19) we note that $l_q(s,x|\chi)$ is an analytic function for $s\in\mathbb{C}$ when $\chi\neq 1$. Furthermore, for each $n\in\mathbb{Z}_+$, we have

$$l_q(-n,x|\chi) = E_{n,\chi,q}(x).$$

In this section we introduced some basic facts about one-variable Euler q–l-series and two-variable Euler q–l-series. Then their values at negative integers are given in terms of generalized q-Euler numbers and polynomials attached to χ. We now evaluate $l_{q,E}(1,x|\chi)$ and give some relation with q-Euler polynomials and numbers. By the definition of $l_{q,E}(s,x|\chi)$, we easily see that

$$l_{q,E}(s,x|\chi)=\frac{[2]_q}{[2]_{q^F}}\sum_{a=1}^{F}(-1)^a\chi(a)\left\{\frac{[2]_{q^F}}{2}[a+x]_q^{-s}+[a+x]_q^{-s}\sum_{j=1}^{\infty}\binom{-s}{j}E_{j,q^F}q^{j(a+x)}\left(\frac{[F]_q}{[a+x]_q}\right)^j\right\}.$$

From the Taylor expansion at $s=0$, we note that

$$[a+x]_q^{-s}=1-s\log[a+x]_q+\cdots.$$

Thus, we have

$$l_{q,E}(0,x|\chi)=\frac{[2]_q}{2}\sum_{a=1}^{F}(-1)^a\chi(a)$$

and

$$l_{q,E}(1,x|\chi)=\frac{[2]_q}{[2]_{q^F}}\sum_{a=1}^{F}\frac{(-1)^a}{[a+x]_q}\left\{\frac{[2]_{q^F}}{2}+\sum_{j=1}^{\infty}(-1)^jE_{j,q^F}q^{j(a+x)}\left(\frac{[F]_q}{[a+x]_q}\right)^j\right\}.$$

In the case $x=0$, we see that

$$l_{q,E}(1,0|\chi)=l_{q,E}(1,\chi)=\frac{[2]_q}{[2]_{q^F}}\sum_{a=1}^{F}\left\{\frac{[2]_{q^F}}{2}+\sum_{j=1}^{\infty}(-1)^jE_{j,q^F}q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^j\right\}.$$

3. p-Adic interpolating function for q-Euler numbers

We shall consider the p-adic analogues of two-variable Euler q–l-functions which are introduced in the previous section. In this section we assume that $q\in\mathbb{C}_p$ with $|1-q|_p<p^{-\frac{1}{p-1}}$ so that $q^x=\exp(x\log q)$. Throughout this section we assume that p is an odd prime. Let ω denote the Teichmüller character having conductor p. For an arbitrary character χ, we define $\chi_n=\chi\omega^{-n}$ where $n\in\mathbb{Z}$, in the sense of the product of characters. Let $\langle a\rangle=\langle a:q\rangle=\omega^{-1}(a)[a]_q=\frac{[a]_q}{\omega(a)}$. Then $\langle a\rangle\equiv 1 \pmod{p^{1+\frac{1}{p-1}}}$. For our purpose, we extend this by defining

$$\begin{aligned}\langle a+pt\rangle&=\omega^{-1}(a+pt)[a+pt]_q=\omega^{-1}(a)[a]_q+\omega^{-1}(a)q^a[pt]_q\\&\equiv 1\quad\bigl(\operatorname{mod} p^{1+\frac{1}{p-1}}\bigr),\quad\text{where } t\in\mathbb{C}_p \text{ with } |t|_p\leqslant 1.\end{aligned}$$

The p-adic logarithmic function, $\log_p$, is the unique function $\mathbb{C}_p^*\to\mathbb{C}_p$ that satisfies

$$\begin{aligned}&\log_p(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^n,\quad |x|_p<1,\\&\log_p(xy)=\log_p x+\log_p y,\quad \forall x,y\in\mathbb{C}_p^*,\quad\text{and}\\&\log_p p=0.\end{aligned}$$

Let

$$A_j(x)=\sum_{j=0}^{\infty}a_{n,j}x^n,\quad a_{n,j}\in\mathbb{C}_p,\ j=0,1,2,\ldots,$$

be a sequence of power series, each of which converges in a fixed subset $D=\{s\in\mathbb{C}_p \mid |s|_p\leqslant p^{1-\frac{1}{p-1}}\}$ of $\mathbb{C}_p$ such that

(1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for $\forall n$, j and
(2) for each $s \in D$ and $\epsilon > 0$, there exists $n_0 = n_0\ (s, \epsilon)$ such that

$$\left| \sum_{n \geqslant n_0} a_{n,j} s^n \right|_p < \epsilon, \quad \text{for } \forall j.$$

Then $\lim_{j\to\infty} A_j(s) = A_0(s)$ for all $s \in D$, see [11–13,15,16].

Let χ be the Dirichlet's character with conductor d (= odd) and let F be a positive integral multiple of p and d. Now we set

$$l_{p,q}(s,t|\chi) = \frac{[2]_q}{[2]_{q^F}} \sum_{\substack{a=1\\ p\nmid a}}^{F} (-1)^a \chi(a) \langle a+pt\rangle^{-s} \sum_{n=0}^{\infty} \binom{-s}{n} E_{n,q^F} q^{n(a+pt)} \left[\frac{F}{a+pt}\right]_{q^{a+pt}}^n. \tag{20}$$

Then $l_{p,q}(s,t|\chi)$ is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leqslant 1$, provided $s \in D$. For $t \in \mathbb{C}_p$ with $|t|_p \leqslant 1$, we have

$$\sum_{j=0}^{\infty} \binom{s}{j} E_{j,q^F} q^{j(a+pt)} \left[\frac{F}{a+pt}\right]_{q^{a+pt}}^j$$

is analytic for $s \in D$. It readily follows that

$$\langle a+pt\rangle^s = \omega^{-s}(a)[a+pt]_q^s = \langle a\rangle^s \sum_{m=0}^{\infty} \binom{s}{m} \left(q^a [a]_q^{-1} [pt]_q\right)^m$$

is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leqslant 1$ when $s \in D$. Thus, we note that

$$l_{p,q}(0,t|\chi) = \frac{[2]_q}{2} \sum_{a=1}^{F} (-1)^a \chi_n(a).$$

We now let $n \in \mathbb{Z}_+$ and fix $t \in \mathbb{C}_p$ with $|t|_p \leqslant 1$. Then we see that

$$E_{n,\chi_n,q}(pt) = [F]_q^n \frac{[2]_q}{[2]_{q^F}} \sum_{a=0}^{F} \chi_n(a)(-1)^a E_{n,q^F}\left(\frac{a+pt}{F}\right).$$

If $\chi_n(p) \neq 0$, then $(p, d_{\chi_n}) = 1$, so that F/p is a multiple of d_{χ_n}. Therefore we have

$$\begin{aligned}
\chi_n(p)[p]_q^n E_{n,\chi_n,q^p}(t) &= \chi_n(p)[p]_q^n \left\{ [F/p]_{q^p}^n \frac{[2]_{q^p}}{[2]_{q^{pF/p}}} \sum_{a=0}^{F/p-1} \chi_n(a)(-1)^a E_{n,(q^p)^{F/p}}\left(\frac{a+t}{F/p}\right) \right\} \\
&= [F]_q^n \frac{[2]_{q^p}}{[2]_{q^F}} \sum_{\substack{a=0\\ p|a}}^{F} \chi_n(a)(-1)^a E_{n,q^F}\left(\frac{a+pt}{F}\right).
\end{aligned}$$

Thus, we note that

$$\frac{[2]_q}{[2]_{q^p}} \chi_n(p)[p]_q^n E_{n,\chi_n,q^p}(t) = \frac{[2]_q}{[2]_{q^F}} [F]_q^n \sum_{\substack{a=0\\ p|a}}^{F} \chi_n(a)(-1)^a E_{n,q^F}\left(\frac{a+pt}{F}\right). \tag{21}$$

The difference of these quantities yields

$$E_{n,\chi_n,q}(pt) - \frac{[2]_q}{[2]_{q^p}} \chi_n(p)[p]_q^n E_{n,\chi_n,q^p}(t) = \frac{[2]_q}{[2]_{q^F}} [F]_q^n \sum_{\substack{a=0\\ p\nmid a}}^{F-1} \chi_n(a)(-1)^a E_{n,q^F}\left(\frac{a+pt}{F}\right). \tag{22}$$

Using distribution for q-Euler polynomials, we easily see that

$$E_{n,q^F}\left(\frac{a+pt}{F}\right)=[F]_q^{-n}[a+pt]_q^n\sum_{k=0}^{n}\binom{n}{k}q^{(a+pt)k}\left[\frac{F}{a+pt}\right]_{q^{a+pt}}^k E_{k,q^F}. \tag{23}$$

Since $\chi_n(a)=\chi(a)\omega^{-n}(a)$ and for $(a,p)=1$, and $t\in\mathbb{C}_p$ with $|t|_p\leqslant 1$, we have

$$\begin{aligned}
&E_{n,\chi_n,q}(pt)-\frac{[2]_q}{[2]_{q^p}}\chi_n(p)[p]_q^n E_{n,\chi_n,q^p}(t)\\
&\quad=\frac{[2]_q}{[2]_{q^F}}\sum_{a=0}^{F-1}\chi_n(a)(-1)^a E_{n,q^F}\left(\frac{a+pt}{F}\right)\\
&\quad=\frac{[2]_q}{[2]_{q^F}}\sum_{\substack{a=0\\ p\nmid a}}^{F-1}\chi_n(a)\langle a+pt\rangle^n(-1)^a\sum_{k=0}^{\infty}\binom{n}{k}q^{(a+pt)k}\left[\frac{F}{a+pt}\right]_{q^{a+pt}}^k E_{k,q^F}.
\end{aligned} \tag{24}$$

From (20)–(24), we derive

$$E_{n,\chi_n,q}(pt)-\frac{[2]_q}{[2]_{q^p}}\chi_n(p)[p]_q^n E_{n,\chi_n,q^p}(t)=l_{p,q}(-n,t|\chi).$$

Therefore we obtain the following theorem.

Theorem 3. *Let F $(=odd)$ be a positive integral multiple of p and d $(=d_\chi)$, and let*

$$l_{p,q}(s,t|\chi)=\frac{[2]_q}{[2]_{q^F}}\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi(a)(-1)^a\langle a+pt\rangle^{-s}\sum_{m=0}^{\infty}\binom{-s}{m}q^{(a+pt)m}\left[\frac{F}{a+pt}\right]_{q^{a+pt}}^m E_{m,q^F}.$$

Then $l_{p,q}(s,t|\chi)$ is analytic for $t\in\mathbb{C}_p$, $|t|_p\leqslant 1$, provided $s\in D$ when $\chi=1$. Furthermore, for each $n\in\mathbb{Z}_+$, we have

$$l_{p,q}(-n,t|\chi)=E_{n,\chi_n,q}(pt)-\frac{[2]_q}{[2]_{q^p}}\chi_n(p)[p]_q^n E_{n,\chi_n,q^p}(t).$$

Thus, $l_{p,q}(s,0|\chi)=l_{p,q}(s,\chi)$ for all $s\in D$, where $l_{p,q}(s,\chi)$ is p-adic Euler q–l-function, see [5].

We now generalize to two-variable p-adic Euler q–l-function, $l_{p,q}(s,t|\chi)$, by modifying $l_{p,q}(s,\chi)$ which is first defined by the interpolating function

$$H_{p,q}^{(E)}(s,a,F)=\frac{(-1)^a}{[2]_{q^F}}\langle a\rangle^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^j E_{j,q^F},\quad \text{for } s\in\mathbb{Z}_p. \tag{25}$$

From (25), we note that

$$\begin{aligned}
H_{p,q}^{(E)}(-n,a,F)&=\frac{(-1)^a}{[2]_{q^F}}\langle a\rangle^{n}\sum_{j=0}^{n}\binom{n}{j}q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^j E_{j,q^F}\\
&=\frac{(-1)^a}{[2]_{q^F}}\omega^{-n}(a)[F]_q^n E_{n,q^F}\left(\frac{a}{F}\right)\\
&=\omega^{-n}(a)H_{q,E}(-n,a,F),\quad \text{for } n\in\mathbb{Z}_+.
\end{aligned}$$

By using the definition of $H_{p,q}^{(E)}(s,a,F)$, we can express $l_{p,q}(s,t|\chi)$ for all $a\in\mathbb{Z}$, $(a,p)=1$, and $t\in\mathbb{C}_p$ with $|t|\leqslant 1$ as follows:

$$l_{p,q}(s,t|\chi)=\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi(a)H_{p,q}^{(E)}(s,a+pt,F). \tag{26}$$

From the proof of Theorem 3, we note that $H^{(E)}_{p,q}(s, a+pt, F)$ is analytic for $t \in \mathbb{C}_p$, $|t| \leqslant 1$, where $s \in D = \{s \in \mathbb{C}_p \mid |s|_p \leqslant p^{1-\frac{1}{p-1}}\}$. The value of $\frac{\partial}{\partial s} l_{p,q}(s, t|\chi)$ is the coefficient of s in the expansion of $l_{p,q}(s, t|\chi)$ at $s = 0$. Using the Taylor expansion at $s = 0$, we see that

$$\langle a+pt\rangle^{-s} = 1 - s\log\langle a+pt\rangle + \cdots \quad \text{and} \quad \binom{-s}{m} = \frac{(-1)^m}{m}s + \cdots.$$

By employing these expansion and some algebraic manipulations, we evaluate the derivative $\frac{\partial}{\partial s} l_{p,q}(0, t|\chi)$. It follows from the definition of $l_{p,q}(s, t|\chi)$ that

$$l_{p,q}(s, t|\chi) = \frac{[2]_q}{[2]_{q^F}} \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)(-1)^a \langle a+pt\rangle^{-s} \sum_{m=0}^{\infty} \binom{-s}{m} q^{(a+pt)m} \left[\frac{F}{a+pt}\right]^m_{q^{a+pt}} E_{m,q^F}.$$

Thus, we have

$$\frac{\partial}{\partial s} l_{p,q}(s, t|\chi)\Big|_{s=0} = \frac{[2]_q}{[2]_{q^F}} \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)(-1)^a \cdot \left(-\log\langle a+pt\rangle E_{0,q^F} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} q^{(a+pt)m} \left[\frac{F}{a+pt}\right]^m_{q^{a+pt}} E_{m,q^F}\right).$$

Since $\omega(a)$ is a root of unity for $(a, p) = 1$, we have

$$\log_p\langle a+pt\rangle = \log_p(a+pt) + \log_p \omega^{-1}(a) = \log_p(a+pt).$$

Therefore we obtain the following theorem.

Theorem 4. *Let χ be a primitive Dirichlet's character with conductor d $(=$ odd$) \in \mathbb{N}$, and let F $(=$ odd$)$ be a positive integral multiple of p and d. Then for any $t \in \mathbb{C}_p$ with $|t|_p \leqslant 1$, we have*

$$\frac{\partial}{\partial s} l_{p,q}(0, t|\chi) = \frac{[2]_q}{[2]_{q^F}} \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)(-1)^a \sum_{m=1}^{\infty} \frac{(-1)^m}{m} q^{(a+pt)m} \left(\frac{[F]_q}{[a+pt]_q}\right)^m E_{m,q^F} - \frac{[2]_q}{2} \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)(-1)^a \log_p(a+pt).$$

4. Further remarks and observations

The q-Euler numbers can be represented by p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$E_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n q^{-x}\, d\mu_{-q}(x), \qquad E_{n,\chi,q}(x) = \int_X q^{-y}\chi(y)[x+y]_q^n\, d\mu_{-q}(y). \tag{27}$$

From (27), we can derive

$$l_{p,q}(s, t|\chi) = \int_{X^*} \chi(x)\langle x+t\rangle^{-s} q^{-x}\, d\mu_{-q}(x), \quad \text{for } s \in \mathbb{Z}_p. \tag{28}$$

When we take $q = 1$, Eqs. (27) and (28) seem to be interesting formulae.

A p-adic distribution μ on X is defined to be an additive map from the collection of compact open set in X to $\mathbb{Q}_p$. Additivity means $\mu(\bigcup_{k=1}^n u_k) = \sum_{k=1}^n \mu(u_k)$, where $n \geqslant 1$ and $\{u_1, u_2, \ldots, u_n\}$ is any collections of disjoint compact open set in X. Indeed, each $a + p^n\mathbb{Z}_p$ can be represented as the union of compact open subset in X as follows:

$$a + p^n\mathbb{Z}_p = \bigcup_{b=0}^{p-1} \left(a + bp^n + p^{n+1}\mathbb{Z}_p\right).$$

Every map μ from the collection of compact open sets in X to $\mathbb{Q}_p$ for which

$$\mu\left(a + p^n\mathbb{Z}_p\right) = \sum_{k=0}^{p^n-1} \mu\left(a + bp^n + p^{n+1}\mathbb{Z}_p\right)$$

holds whenever $a + p^n\mathbb{Z}_p \subset X$ extends to a p-adic distribution on X. Let p be an odd prime and let k be a positive integer. Now we define a map μ_k on the balls in $\mathbb{Z}_p$ as

$$\mu_k\left(a + p^n\mathbb{Z}_p\right) = (-1)^a p^{nk} f_k\left(\frac{\{a\}_n}{p^n}\right),$$

where $\{a\}_n$ is the unique number in the set $\{0, 1, 2, \dots, p^n - 1\}$ such that $\{a\}_n \equiv a \pmod{p^n}$. If $a \in \{0, 1, 2, \dots, p^n - 1\}$, then

$$\begin{aligned}
\sum_{b=0}^{p-1} \mu_k\left(a + bp^n + p^{n+1}\mathbb{Z}_p\right) &= \sum_{b=0}^{p-1} (-1)^{a+bp^n} p^{(n+1)k} f_k\left(\frac{a + bp^n}{p^{n+1}}\right) \\
&= (-1)^a \sum_{b=0}^{p-1} (-1)^b p^{(n+1)k} f_k\left(\frac{a + bp^n}{p^{n+1}}\right) \\
&= (-1)^a p^{nk} \left\{ p^k \sum_{b=0}^{p-1} (-1)^b f_k\left(\frac{\frac{a}{p^n} + b}{p}\right) \right\}.
\end{aligned}$$

From this, we note that μ_k is a distribution on $\mathbb{Z}_p$ if and only if

$$f_k(x) = p^k \sum_{b=0}^{p-1} (-1)^b f_k\left(\frac{x + b}{p}\right). \tag{29}$$

Let μ_k be defined as

$$\mu_k\left(a + p^n\mathbb{Z}_p\right) = (-1)^a p^{nk} E_k\left(\frac{a}{p^n}\right).$$

By (29), we see that μ_k is a distribution on $\mathbb{Z}_p$. A p-adic distribution μ on X is called a p-adic measure on X if its values on compact open subsets $U \subset X$ are bounded: there exists some constant M such that $|\mu(U)|_p \leqslant M$ for all compact open subsets $U \subset X$. It is easy to see that

$$\mu_k\left(a + p^n\mathbb{Z}_p\right) \equiv (-1)^a a^k \pmod{p^n},$$

so, $|\mu_k| \leqslant M$ for some constant M. Hence, p-adic Euler distribution becomes a p-adic measure on $\mathbb{Z}_p$. By the definition of p-adic Euler measure, we see that

$$\int_{\mathbb{Z}_p} d\mu_k(x) = \int_{\mathbb{Z}_p} x^k \, d\mu_{-1}(x) = E_k.$$

References

[1] L.C. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948) 987–1000.

[2] M. Cenkci, M. Can, V. Kurt, p-Adic interpolation functions and Kummer-type congruences for q-twisted Euler numbers, Adv. Stud. Contemp. Math. 9 (2004) 203–216.

[3] H. Ozden, Y. Simsek, S.-H. Rim, I.N. Cangul, A note on p-adic q-Euler measure, Adv. Stud. Contemp. Math. 14 (2007) 124–131.

[4] T. Kim, The modified q-Euler numbers, arXiv: math.NT/0702523.

[5] T. Kim, On p-adic q-l-functions and sums of powers, J. Math. Anal. Appl. 329 (2007) 1472–1481.

[6] T. Kim, On certain values of p-adic q–l-function, Rep. Fac. Sci. Engrg. Saga Univ. Math. 23 (1995) 1–7.
[7] T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 9 (2002) 288–299.
[8] T. Kim, J.Y. Choi, J.Y. Sug, Extended q-Euler numbers and polynomials associated with fermionic p-adic q-integral on $\mathbb{Z}_p$, Russ. J. Math. Phys. 14 (2007) 160–163.
[9] Y. Simsek, Twisted (h,q)-Bernoulli numbers and polynomials related to twisted (h,q)-zeta function and L-function, J. Math. Anal. Appl. 324 (2006) 790–804.
[10] Y. Simsek, Theorems on twisted L-function and twisted Bernoulli numbers, Adv. Stud. Contemp. Math. 11 (2005) 205–218.
[11] K. Shiratani, S. Yamamoto, On p-adic interpolating function for the Euler numbers and its derivatives, Mem. Fac. Sci. Kyushu Univ. Ser. A 39 (1985) 113–125.
[12] L. Washington, Introduction to Cyclotomic Fields, Springer-Verlag, 1982.
[13] C.F. Woodcock, Special p-adic analytic functions and Fourier transformations, J. Number Theory 60 (1996) 393–408.
[14] T. Kim, A note on p-adic q-integrals associated with q-Euler numbers, arXiv: 0706.4341, preprint, 2007, 5 pp.
[15] M. Cenkci, Y. Simsek, V. Kurt, Further remarks on multiple p-adic q–L-function of two variables, Adv. Stud. Contemp. Math. 14 (2007) 49–68.
[16] M. Cenkci, M. Can, Some results on q-analogue of the Lerch zeta function, Adv. Stud. Contemp. Math. 12 (2006) 213–223.